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Exponential Windfields

by

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§ 1 Introduction

In this report it will be investigated in how far actual storms may be represented by simple mathematical expressions of a certain kind. We shall consider storms which have the same intensity, and direction throughout the sea, and which are characterised by a single rise and fall. We shall try to represent the surface stress S at the sea level by an expression of the kind

$$S = \sum C_k e^{p_k t} \qquad 1.1$$

with $p_k > 0$.
The surface stress is proportional to the square of the velocity of the wind at sea level. The duration of the storm may be defined as the time during which the velocity of the wind exceeds half its maximum value, i.e. during which $|S| > 1/4 \, |S_{max}|$ [1]. For actual storms which may be dangerous this period is of the order of one day.
The formula 1.1 will be tested upon the mathematical model of an infinitely wide sea of a constant depth [2]. In this model we have the following differential equations

$$\begin{cases} \left(\frac{\partial}{\partial t} + \lambda\right) u - \Omega v + g \frac{\partial \zeta}{\partial x} = \frac{U}{\rho h} \\ \left(\frac{\partial}{\partial t} + \lambda\right) v + \Omega u + g \frac{\partial \zeta}{\partial y} = \frac{V}{\rho h} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{h} \frac{\partial \zeta}{\partial t} = 0 , \end{cases} \qquad 1.2$$

and the boundary conditions

$$\begin{array}{lll} \text{coast} & y = 0 & v = 0, \\ \text{ocean} & y = b & \zeta = 0. \end{array} \qquad 1.3$$

Here g is the acceleration of gravity, λ a friction coefficient, Ω the coefficient of Coriolis, h the depth, u and v averages over a vertical of the horizontal components of the velocity, ζ the elevation of the sealevel above the undisturbed level, U and V the components of the tangential stress on the surface of the sea due

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1) The concept "storm" is not used here in the meteorological sense.
2) cf. TW 41.

to the wind.
In accordance with the actual conditions of the North Sea we shall take the following numerical values.

$\lambda = 0,09\ h^{-1}$ $\Omega = 0.48\ h^{-1}$ (mean value)

$b = 850$ km $h = 65$ m (harmonic average)

Next, if W represents the velocity of the wind at sea level

$$S = \frac{\sqrt{U^2+V^2}}{\rho} = 3.5 \times 10^{-6}\ W^2$$

in corresponding units.
The following units will now be introduced in 1.2 and 1.3

t	$b/2\pi\sqrt{gh}$	$= 1.5$ h
x,y	$b/2\pi$	$= 135$ km
u,v	$\sqrt{gh}$	$= 91$ km/h
ζ	h	$= 65$ m

Then 1.2 and 1.3 become for a situation in which u,v and ζ are independent of x

$$\left\{\begin{aligned} &\left(\frac{\partial}{\partial t}+\lambda\right)u - \Omega v = U \\ &\left(\frac{\partial}{\partial t}+\lambda\right)v + \Omega u + \frac{\partial \zeta}{\partial y} = V \\ &\frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0\ , \end{aligned}\right. \qquad 1.4$$

$$\left\{\begin{aligned} &y = 0, \quad v = 0 \\ &y = 2\pi, \quad \zeta = 0\ , \end{aligned}\right. \qquad 1.5$$

where $\lambda = 0,14$, $\Omega = 0.71$.
If W is in m/sec, then

$$S = \sqrt{U^2+V^2} = 1.1 \times 10^{-5}\ W^2. \qquad 1.6$$

In the case of the historical storm of February 1953 which caused the flood disaster in the southern parts of Holland and elsewhere we have the following data, according to 1.4, 1.5 and 1.6 (mean values at Hoek van Holland) [1]

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1) Cf. Wemelsfelder. De Ingenieur, 14.8.53, p.372.

time in 1.5 hr	absolute velocity in m/sec	surface stress $\sqrt{U^2+V^2}$	y-component V	x-component U
0	11	1.3×10^{-3}	1.1×10^{-3}	0.7×10^{-3}
5	13	1.9 "	-1.1 "	1.5 "
10	18	3.6 "	-3.1 "	1.8 "
15	25	6.9 "	-6.0 "	3.5 "
20	23	5.8 "	-5.6 "	1.5 "
25	18	3.6 "	-3.5 "	-0.8 "
30	13	1.9 "	-1.8 "	-0.6 "

By the y-component is meant the component along the long side of the North Sea which makes an angle of about 15° with the N-S direction (cf graph 1).

In view of the model considered above only the effect of V will be considered. Thus, in 1.4 we shall take U=0.

It will be tried to represent S, or V, by a formula of the kind 1.1 so that the part of the curve near its maximum resembles that of an actual storm. In section 2 an expression of two terms and one of three terms will be considered. It appears that by two terms a storm may be represented with a slow rise and a steep fall. By an expression of three terms a more symmetrical storm may be represented.

If the expression 1.1 is used for the complete t-range $t > -\infty$ considerable mathematical simplifications are obtained. The solution of 1.4 may be represented by

$$\zeta(y,t) = \sum C_k \ \varphi(y,p_k)e^{p_k t} \ , \qquad 1.7$$

where $\varphi(y,p)$ is obtained by solving the system obtained from 1.4, 1.5 by Laplace transformation.

In formula 1.1 the storm is considered to start at t=0 i.e. $S(o) = 1/4 \ S_{max}$. However, in the mathematical evaluation also the complete past $-\infty < t < 0$ of the storm enters into the consideration.

In view of its practical consequences the elevation at the coast $\zeta(o,t)$ is most important. Its response to a storm is likewise a rise and a fall, however, somewhat lagging behind that of the storm.

In section 3 it will be discussed whether the part of the storm

1.1 has any influence upon the elevation at the coast.
It appears that there is hardly any influence upon the maximum value of $\zeta(o,t)$. This is mainly due to the effect of the friction coefficient λ. However, also the Coriolis coefficient Ω has the effect of an apparent damping.
Therefore the use of formula 1.1 for $t > -\infty$ has been validated.
As concerns the numerical consequences we may add that an expression 1.1 with three terms requires a higher numerical accuracy than one with two terms only etc. In section 4 the effect of the two storms of the type 1.1 upon the elevation at the coast of the model considered above is computed. It appears that the maximum of the elevation occurs a few hours later than the maximum of the storm.
The calculations are also carried out for the case $\Omega = 0$. In this case much higher values of $\zeta(o,t)$ are found. However, this apparent strong influence of the Coriolis-effect might be a characteristic of the particular model.
In a future report the rectangular model of the North Sea $0 < x < \pi$, $0 < y < 2\pi$ will be considered. In this case, however, the influence of the Coriolis-effect may be expected to be less pronounced in view of the guiding effect of the coasts at $x=0,\pi$.

§ 2 Storm formulae

We consider a storm formula 1.1 with two terms

$$S = C_1 e^{p_1 t} - C_2 e^{p_2 t} \qquad 2.1$$

with $p_2 > p_1$.

If t and S are measured in suitable new units then 2.1 becomes

$$S_o = e^{t_o} - e^{\theta t_o} \qquad 2.2$$

with $\theta > 1$.
The maximum of S_o is $S_m = (\theta - 1)\theta^{-\frac{\theta}{\theta-1}}$ and is reached at $t_m = -\frac{1}{\theta-1} \ln \theta$.
We shall take the numerical case $\theta = 3/2$, $p_1 = 0.12$, $p_2 = 0.18$. Then 2.1 becomes for $C_1 = 1$

$$S = e^{0.12t} - 0.2176\, e^{0.18t} \qquad 2.3$$

with

$$S_m = 3.13 \quad \text{for} \quad t_m = 18.5.$$

For a few values of t 2.3 becomes

t = 0	S =	0.78
5		1.29
10		2.00
15		2.81
20		3.06
25		0.46
30		-11.56

The duration of this storm is about 25 time units i.e. about $37\frac{1}{2}$ hr. In order to make this storm comparable to that of an actual North Sea storm to 2.3 the factor 2×10^{-3} may be added. Thus we have e.g.

$$V = -2 \times 10^{-3} \left(e^{0.12t} - 0.2176\, e^{0.18t}\right). \qquad 2.4$$

This storm (cf graph 2) starts at t=0 and has a maximum of 24 m/sec at t=18.5 and stops at t=24.5.
This storm is characterized by a rather slow rise and a rather steep fall.
By adding a third exponential term a greater variety of storm curves is possible and more symmetrical storms may be obtained.
We shall consider the formula

$$S = C_1 e^{p_1 t} - C_2 e^{p_2 t} + C_3 e^{p_3 t} \qquad 2.5$$

with $p_3 > p_2 > p_1$.
If in particular $p_3 = 3/2\, p_2$, $p_2 = 3/2\, p_1$ formula 2.5 is equivalent to

$$S_o = u - u^{3/2} + \gamma u^{9/4} , \quad u = e^{t_o} . \qquad 2.6$$

S_o may have two zeros between which $S_o < 0$. For $\gamma = \frac{6}{125}\sqrt{15} = 0.186$ both zeros coincide and hence $S_o > 0$ for all $u > 0$.
From this we may obtain the following formula

$$S = e^{0.12t} - 0.2570 e^{0.18t} + 0.006223\, e^{0.27t} \qquad 2.7$$

with

$$S_m = 3.00 \quad \text{for} \quad t_m = 20.$$

For a few values of t 2.7 becomes

t = 0	S = 0.75
5	1.21
10	1.86
15	2.58
20	3.00
25	2.37
30	0.20
32	0.16

The duration of this storm is about 29 time units or 43½ hr. If the factor 2×10^{-3} is added we obtain

$$V = -2 \times 10^{-3} \left(e^{0.12t} - 0.2570\, e^{0.18t} + 0.006223\, e^{0.27t}\right) \qquad 2.8$$

This storm starts at t=0, has a maximum of 23 m/sec at t=20 and stops at t=29 (cf graph 3).

§ 3 The aftereffect of a disturbance

We shall consider the aftereffect of a disturbance of the sea. We shall take the model of the infinitely wide North Sea mentioned before, however, with $\Omega = 0$ in order to avoid mathematical complications. The equations are according to 1.4

$$\begin{cases} \left(\frac{\partial}{\partial t} + \lambda\right) v + \frac{\partial \zeta}{\partial y} = V \\ \frac{\partial v}{\partial y} + \frac{\partial \zeta}{t} = 0 , \end{cases} \qquad 3.1$$

with

$$\begin{cases} y = 0 \qquad v = 0 , \\ y = 2 \qquad \zeta = 0 . \end{cases} \qquad 3.2$$

A uniform and constant wind V=-1 effects the stationary situation

$$v = 0, \qquad \zeta = 2\pi - y.$$

If the wind suddenly stops at t=0 the aftereffect of the wind may

be calculated as follows.
From 3.1 and 3.2 we obtain after Laplace transformation

$$\bar{\zeta}(y,\rho) \overset{def}{=} \int_0^\infty e^{-pt}\,\zeta(y,t)\,dt \qquad 3.3$$

etc.

$$\frac{\partial^2\bar{\zeta}}{\partial y^2} - p(p+\lambda)\bar{\zeta} = -(p+\lambda)(2\pi - y), \qquad 3.4$$

with

$$\begin{cases} y = 0 & \dfrac{\partial\bar{\zeta}}{\partial y} = 0 \\ y = 2 & \bar{\zeta} = 0. \end{cases}$$

The solution is

$$\bar{\zeta}(y,p) = \frac{2\pi - y}{p} \cdot \frac{\sinh(2\pi - y)q}{pq\,\cosh 2\pi q}, \qquad 3.5$$

with $q^2 = p(p+\lambda)$.

At y=0 we have in particular

$$\bar{\zeta}(0,p) = \frac{2\pi}{p} - \frac{\tanh 2\pi q}{pq}. \qquad 3.6$$

From 3.6 we obtain (cf TW 41 section 8)

$$\zeta(0,t) = 2\pi - \sum_0 (-1)^j\,\varepsilon_j \int_{4j\pi}^{t} e^{-\frac{\lambda}{2}\tau}\, I_o\!\left(\tfrac{\lambda}{2}\sqrt{\tau^2 - 16j^2\pi^2}\right) d\tau. \qquad 3.7$$

where $\varepsilon_j=1$ for $j=0$ and $\varepsilon_j=2$ for $j \geq 1$.
The summation in 3.7 breaks off at $j=\left[\frac{t}{4\pi}\right]$.
By means of 3.7 $\zeta(0,t)$ has been calculated for a few t-values.
The results are

t	$\zeta(0,t)$
0	6.28
2	4.41
4	2.78
6	1.32
8	0.03
10	-1.14
15	-1.73
20	-0.32
25	0.94

30	0.47
35	-0.07
40	-0.30
50	0.13

We see that after an interval of 20 units of time (30 h) the aftereffect is relatively small and of the order of 15%. For the part of the storms 2.4 and 2.8, i.e. for $t < 0$, we may expect an effect which is four times smaller than that at the time of the maximum intensity.
In other words, the influence of the past of these storms $(t < 0)$ upon the maximum elevation at the coast may be expected to be of the order of 4%.

§4. Two examples

In this section we shall determine the effect of the storms 2.4 and 2.8 upon the elevation at the coast. If

$$V = \sum c_k \, e^{p_k t} , \qquad 4.1$$

the solution of 1.4 and 1.5 is of the form

$$\zeta(y,t) = \sum c_k \, \varphi(y,p_k) e^{p_k t} . \qquad 4.2$$

For arbitrary p we find easily

$$\varphi(y,p) = - \frac{\sinh(2\pi - y)q}{q \cosh 2\pi q} , \qquad 4.3$$

where

$$q^2 = p(p+\lambda) + \Omega^2 \frac{p}{p+\lambda} .$$

Thus we have at y=0

$$\zeta(0,t) = \sum c_k \frac{\tanh 2\pi q_k}{q_k} e^{p_k t} . \qquad 4.4$$

The windfields are resp.

$$V_1 = -2 \times 10^{-3} \left(e^{0.12t} - 0.2176 \, e^{0.18t}\right). \qquad 4.5$$

$$V_2 = -2 \times 10^{-3} \left(e^{0.12t} - 0.2570 \, e^{0.18t} + 0.006223 \, e^{0.27t}\right). \qquad 4.6$$

From 4.4 the following table is obtained. We have for ζ in meters

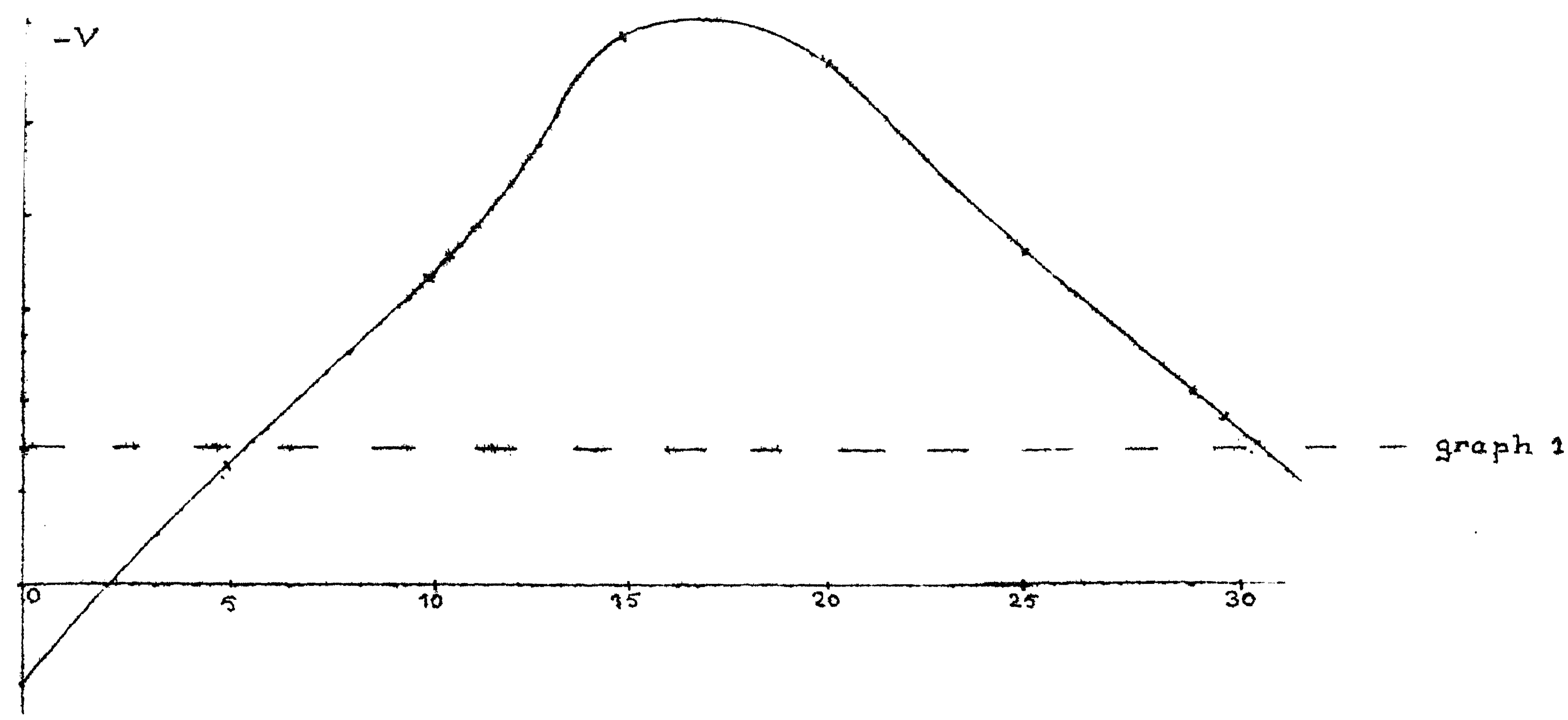
t = 0	$\zeta_1(0,t)$ = 0.21	$\zeta_2(0,t)$ = 0.20
5	0.34	0.32
10	0.55	0.51
15	0.81	0.75
20	1.02	0.97
25	0.72	0.97
30	-1.48	0.62

In the first case we obtain a maximum at the coast of 1.02 m. which is reached shortly (about 3 hours) after the maximum of the storm; in the second case a maximum of 0.97 m. is obtained with a similar delay.
The aftereffect of the situation at t=0 when the sea is only slightly disturbed may be expected to be small at the time of the maximum elevation at the coast. According to the results of the preceding section this effect is of the order of a few percent only. A more careful analysis proves this to be of the order of 4%.

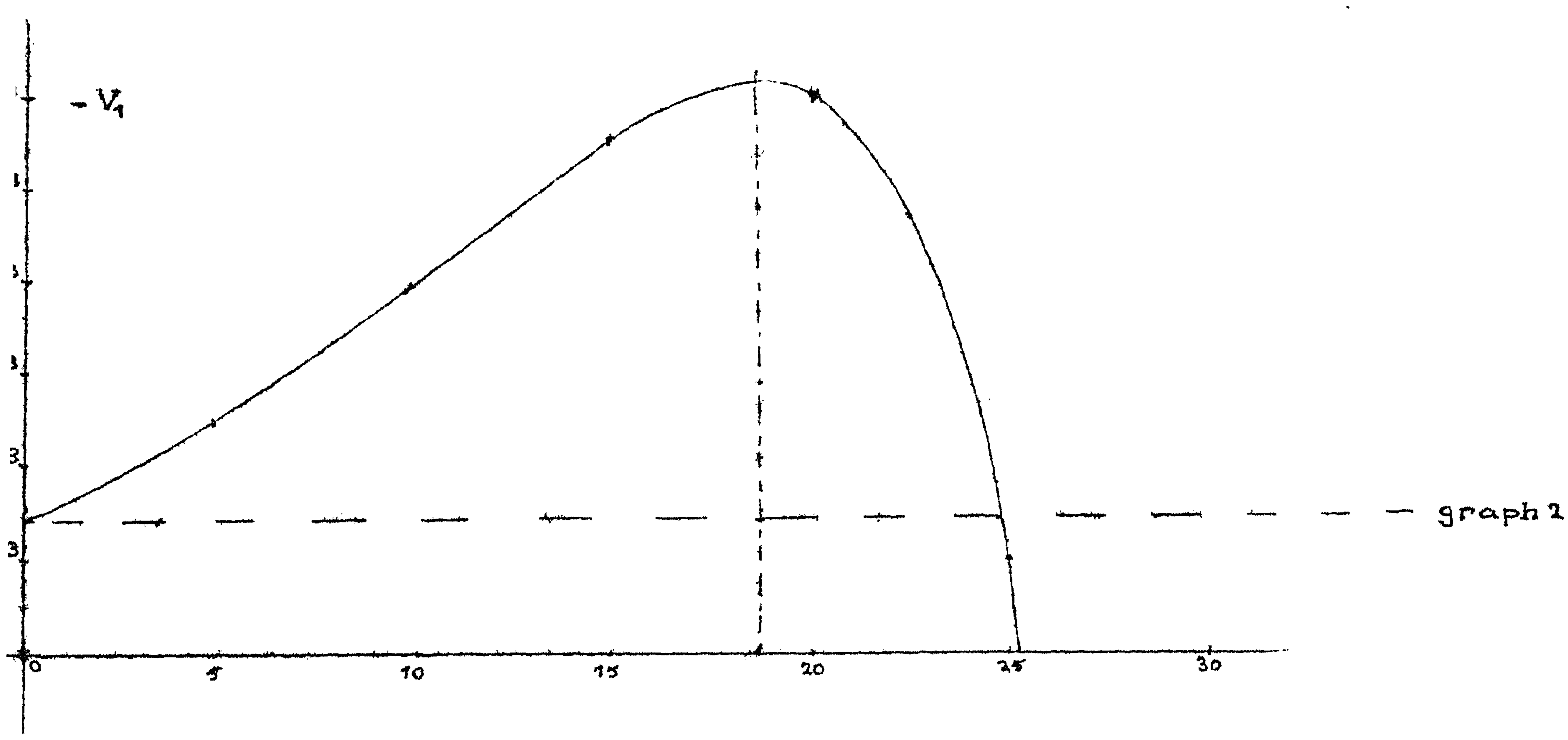
If in the preceding calculation the coefficient of Coriolis is taken zero, $\Omega=0$, considerably higher values of ζ are obtained. We have for ζ in meters

	$\zeta_1(0,t)$ $\Omega = 0$	$\zeta_2(0,t)$
t = 0	0.49	0.47
5	0.81	0.77
10	1.31	1.23
15	1.99	1.83
20	2.61	2.42
25	2.26	2.54
30	-2.00	-1.48

I

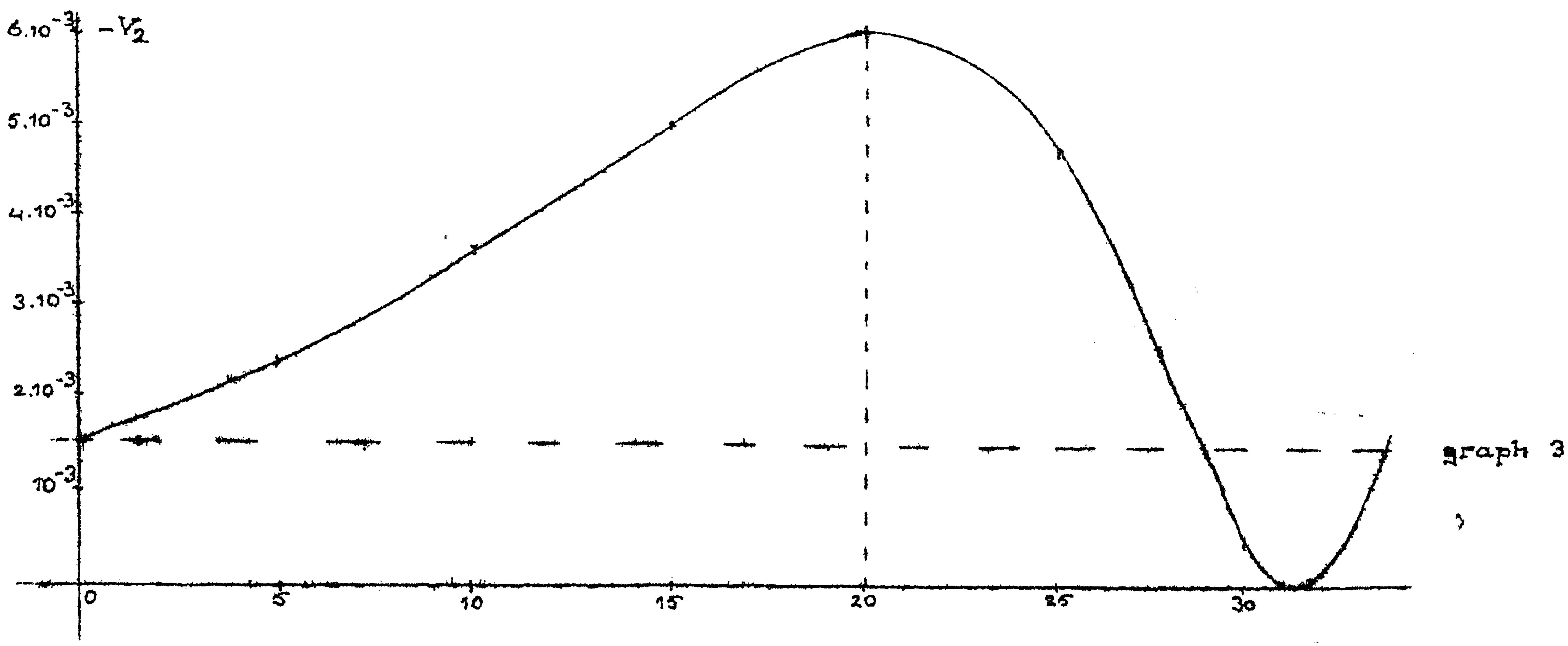


al component of surface stress due to the storm of February 1953.



$$V_1 = -2.10^{-3}\left(e^{0.12t} - 0.2176\, e^{0.18t}\right).$$

II



$$V_2 = -2.10^{-3}\left(e^{0.12t} - 0.257\, e^{0.18t} + 0.006223\, e^{0.27t}\right)$$